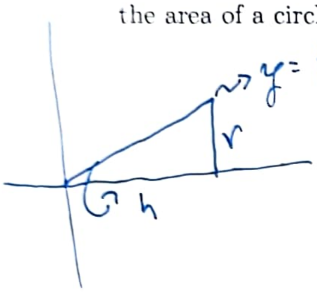
Solutions

April 29, 2025

Math 1A Worksheet #35

Name: ______________________________

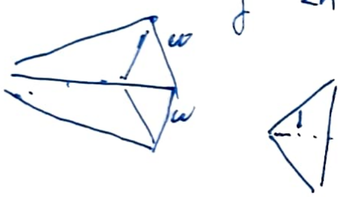
1. The formula for the volume of a cone with height h and base radius r is $\frac{1}{3}h\pi r^2$. Given just the formula for the area of a circle, can you use calculus to prove that this formula for the cone volume is correct?



$y = \frac{r}{h}x$

$$\int_0^h \left(\frac{r}{h}x\right)^2 \cdot \pi\, dx = \int_0^h \frac{\pi r^2}{h^2} x^2\, dx = \frac{1}{3}\frac{\pi r^2}{h^2} x^3 \Big|_0^h = \frac{1}{3}\pi r^2 h$$

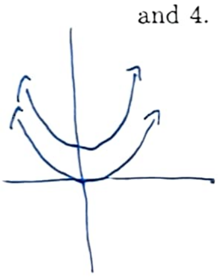
2. If a pyramid with a square $w \times w$ base has height h, it's volume will be $\frac{1}{3}hw^2$. Note how similar this is to the formula for the volume of a cone. Can you prove this formula is correct using calculus?



$y = \frac{w}{2h}x$

$$\int_0^h \left(2 \cdot \frac{w}{2h}x\right)^2 dx = \frac{1}{3}hw^2$$

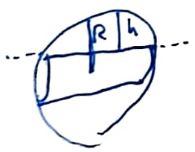
3. I want to make a bowl by rotating the curves $y = x^2$ and $y = x^2 + 1$ about the y-axis, for y in between 0 and 4. How much clay do I need?



$$\int_0^4 \pi\left[(\sqrt{y})^2 - (\sqrt{y-1})^2\right] dy = \int_0^4 \pi\left[y - y + 1\right] dy = 4\pi$$

4. A napkin ring is a sphere with a cylinder cut right through the middle. See 5.3.48 for a picture. Lets say the sphere has radius R and after you cut out the cylinder, the height of the remaining ring is h. How much material is used by the napkin ring, in terms of R and h? What's surprising about the answer?

lazy



$y^2 + x^2 = R^2$

$x = \sqrt{R^2 - (R-h)^2}$

$$2\int_0^{\sqrt{R^2-(R-h)^2}} \pi(R^2 - x^2)\, dx = 2\pi\left(R^2 x - \frac{1}{3}x^3\right)\Big|_0^{-}$$

$$= 2\pi R^2\sqrt{R^2-(R-h)^2} - \frac{2\pi}{3}\left(\sqrt{R^2-(R-h)^2}\right)^3$$

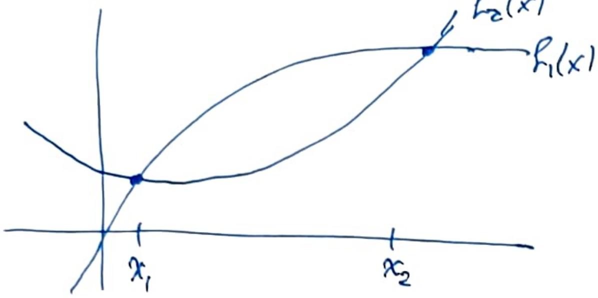
5. I make an infinitely long horn by rotating the curve $y = 1/x$ for $x \geq 1$ about the x-axis. How much air is in this horn? What's the surface area? What's surprising about these answers?

Vol. $\int_1^\infty \pi \cdot \frac{1}{x^2}\, dx = -\frac{\pi}{x}\Big|_1^\infty = \pi$

S.A. $\int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1 + \frac{1}{x^2}}\, dx = \int_1^\infty 2\pi \frac{\sqrt{x^2+1}}{x^2}\, dx$ - trig integral...

Mini-Lecture

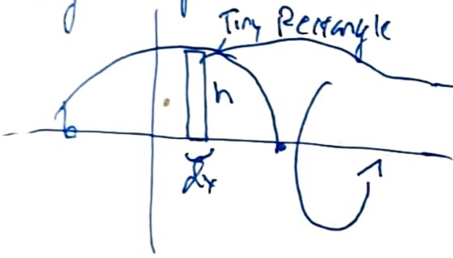
1.) Areas B/t Curves



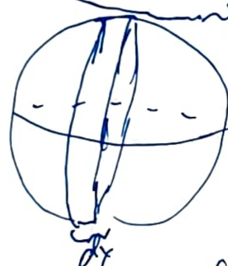
$$\int_{x_1}^{x_2} |f_1(x) - f_2(x)| \, dx$$

2.) Volumes

e.g. : Sphere - Rotation of circle



- We view volume $\approx$ area $\cdot$ height



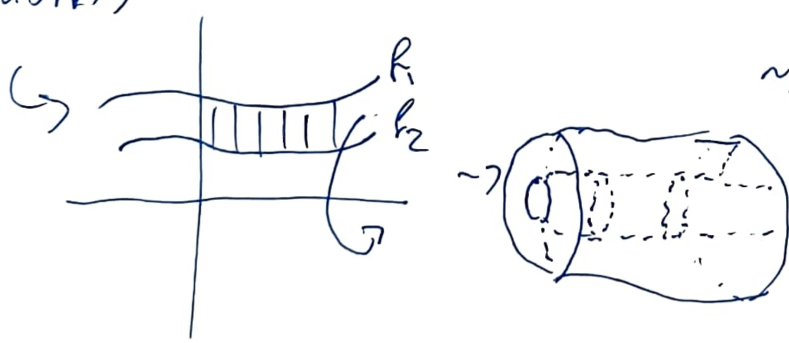
$\rightsquigarrow$ tiny Cylinder

Vol. $= \pi (h)^2 dx$

integrate $\int \pi h^2 dx \rightsquigarrow \int_{-R}^{R} \pi \left(\sqrt{R^2 - x^2}\right)^2 dx = \int_{-R}^{R} \pi R^2 - \pi x^2 dx$

$= 2\pi R^3 - \frac{2}{3}\pi R^3 = \frac{4}{3}\pi R^3$

washers



~ like areas b/t curves - subtract

$$\int \pi (f_1)^2 - \pi (f_2)^2 \, dx$$